Part 6: Maximum Variance

To determine the maximum theoretical standard variance from an average value specified for a value-bounded set, the squares must be maximized. If the range of values is from p_a to p_b , and the average value is m , the maximum variance can be calculated fairly simply.

The largest variances possible will be found where the largest possible gaps are. These are from p_a to $m = m - p_a$, and from m to $p_b = p_b - m$. The values attained from each of these gaps must contribute to an average value of m . To do this, we require:

$$m = \frac{q_a p_a + q_b p_b}{q_a + q_b}$$

Where q_a is the quantity of p_a and q_b is the quantity of p_b .

The standard variance will then be:

$$\sigma_{max}^2 = \frac{q_a (m - p_a)^2 + q_b (p_b - m)^2}{q_a + q_b}$$

$$\sigma_{max}^2 = \frac{q_a (m^2 + p_a^2 - 2 m p_a) + q_b (p_b^2 + m^2 - 2 m p_b)}{q_a + q_b}$$

$$\sigma_{max}^2 = \frac{(q_a + q_b) m^2}{(q_a + q_b)} + \frac{q_a p_a^2 + q_b p_b^2}{(q_a + q_b)} - 2m \frac{(q_a p_a + q_b p_b)}{q_a + q_b}$$

Simplifying the first term in this equation and substituting the previous equation into it, we obtain:

$$\sigma_{max}^2 = m^2 + \frac{q_a p_a^2 + q_b p_b^2}{(q_a + q_b)} - 2m^2$$

$$\boxed{\sigma_{max}^2 = \frac{q_a p_a^2 + q_b p_b^2}{(q_a + q_b)} - m^2}$$

Again, from the first equation we can directly derive:

$$q_a p_a = m (q_a + q_b) - q_b p_b$$

And its counterpart:

$$q_b p_b = m (q_a + q_b) - q_a p_a$$

Substituting these into our working equation, we have:

$$\sigma_{max}^2 = \frac{p_a(m(q_a+q_b)-q_b p_b)+p_b(m(q_a+q_b)-q_a p_a)}{(q_a+q_b)} - m^2$$

$$\sigma_{max}^2 = \frac{m(q_a+q_b)(p_a+p_b)}{(q_a+q_b)} - \frac{(q_a+q_b)(p_a p_b)}{(q_a+q_b)} - m^2$$

$$\sigma_{max}^2 = m(p_a+p_b) - p_a p_b - m^2$$

$$\sigma_{max}^2 = m(p_b-m) - p_a(p_b-m)$$

$$\boxed{\sigma_{max}^2 = (m-p_a)(p_b-m)}$$

Which is simply the product of the two maximum gaps.

The maximum standard variance for the population M, where $F(m_{avg})$ is the average fitness of the population and the maximum fitness is $F(m)_{max}$ will therefore be:

$$\boxed{\sigma_{max}^2 = F(m_{avg})(F(m)_{max} - F(m_{avg}))}$$

Therefore, the maximum possible average expected value after natural selection must always be:

$$F_e(m'_{avg})_{max} = F(m_{avg}) + \frac{\sigma_{max}^2}{F(m_{avg})}$$

$$F_e(m'_{avg})_{max} = F(m_{avg}) + \frac{F(m_{avg})(F(m)_{max} - F(m_{avg}))}{F(m_{avg})}$$

$$F_e(m'_{avg})_{max} = F(m_{avg}) + F(m)_{max} - F(m_{avg})$$

$$\boxed{F_e(m'_{avg})_{max} = F(m)_{max}}$$

Which should be expected, and therefore verifies the above equations.